FSBday:
Implementing Wagner’s Generalized Birthday Attack against the round-1 SHA-3 Candidate FSB

Dan Bernstein, Tanja Lange, Ruben Niederhagen, Christiane Peters and Peter Schwabe

Eindhoven University of Technology
University of Illinois at Chicago
RWTH Aachen University

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Wagner’s generalized birthday attack

Given $2^{i-1}$ lists containing $B$-bit strings.

**Generalized birthday problem:**
The $2^{i-1}$-sum problem consists of finding $2^{i-1}$ elements—exactly one per list—such that their sum equals 0 (bitwise modulo $2 \Rightarrow \text{xor}$).

**Wagner (CRYPTO ’02)**
We can expect a solution to the generalized birthday problem after one run of an algorithm using time $O((i - 1) \cdot 2^{B/i})$ and lists of size $O(2^{B/i})$. 
Wagner’s tree algorithm

1. merge

2. merge

$L_{0,0}$  $L_{0,1}$  $L_{0,2}$  $L_{0,3}$  $2^{B/3}$ elements

compare on least significant $B/3$ bits

$L_{1,0}$  $L_{1,1}$  $2^{B/3}$ elements of xors

compare on least significant $2 \times B/3$ bits

$L_{2,0}$

Expect to get 1 match after the last merge step.
Tree algorithm for $2^{i-1}$ lists

The tree algorithm generalizes to $2^{i-1}$ lists as follows:

- Compare lists — always two at a time — by looking at the least significant $B/i$ bits of elements.

- On level $i - 2$ we are left with two lists whose elements need to be compared on $2B/i$ remaining bits.
Precomputation step

Suppose that there is space for lists of size only $2^c$ with $c < B/i$.

Bernstein (SHARCS '07):

- Generate $2^{c \cdot (B - ic)}$ entries and only consider those of which the least significant $B - ic$ bits are zero.

- Then apply Wagner's algorithm with lists of size $2^c$ and clamp away $c$ bits on each level.

**Wagner**

\[
\begin{array}{ccccccc}
B/i & B/i & B/i & \cdots & B/i \\
\end{array}
\]

**Bernstein**

\[
\begin{array}{ccccccc}
c & c & c & \cdots & B - ic \\
\end{array}
\]

\[\begin{array}{cccc}
B
\end{array}\]
Repeating (parts of) the tree algorithm

- When performing the algorithm with smaller lists, \( u \) bits remain “uncontrolled” at the end.

- Deal with the lower success probability by repeatedly running the attack with different clamping values.

\[
\begin{align*}
\text{Wagner} & \quad \begin{array}{cccc|} 
B/i & B/i & B/i & \cdots & B/i \\
\end{array} \\
\text{Bernstein} & \quad \begin{array}{cccc|} 
\phantom{B/i} & \phantom{B/i} & \phantom{B/i} & \cdots & B - ic \\
\end{array} \\
\text{uncontrolled bits} & \quad \begin{array}{cccc|} 
u & c' & c' & c' & \cdots & B - ic' - u \\
\end{array}
\end{align*}
\]
Target: the compression function of FSB$_{48}$

Given a binary random $192 \times 393216$ matrix $H$; number of blocks:

$w = 24$.

**Input:** a regular weight-24 bit string of length 393216, i.e., there is exactly a single 1 in each interval $[(i - 1) \cdot 16384, i \cdot 16834]_{1 \leq i \leq 24}$.

**Output:** Xor the 24 columns indicated by the input bit string.

**Goal:** Find a collision in FSB$_{48}$’s compression function; i.e., find 48 columns—exactly 2 per block—which add up to 0.
Applying Wagner to FSB$_{48}$

Determine the number of lists for a Wagner attack on FSB$_{48}$.

- We choose 16 lists to solve this particular 48-sum problem. (16 is the highest power of 2 dividing 48).

- Each list entry will be the xor of three columns coming from one and a half blocks (no overlaps!).
  $\rightarrow$ We can generate at most $2^{40}$ elements per list.

Straightforward Wagner

- Applying Wagner’s attack with 16 lists in a straightforward way means that we need to have at least $2^\left\lceil 192/5 \right\rceil$ entries per list.

- By clamping away 39 bits in each step we expect to get at least one collision after one run of the tree algorithm.
List entries

- Reduce amount of data by clamping away 2 bits $\Rightarrow 2^{38}$ entries per list (clamp 38 bits on each level)

- Ultimately we are not interested in the value of the entry; but in the column positions in the matrix that lead to this all-zero value.
  - Value-only representation
  - Positions-only representation: keep full positions; if we need the value (or parts of it) it can be dynamically recomputed from the positions.

- Note: Unlike storage requirements for values the number of bytes for positions increases with increasing levels.
Storing positions

- Encode column positions of each entry in 40 bits (5 bytes) for the first level.

- The expected number of entries per list remains the same but the number of lists halves; so the total amount of data is the same on each level when using dynamic recomputation.

- Storing 16 lists with $2^{38}$ entries, each entry encoded in 5 bytes requires **20480 GB** of storage space.

- The Coding and Cryptography Computer Cluster at Eindhoven University of Technology only has a total hard disk space of about 5440 GB, so we **have to adapt our attack strategy** to this limitation.
Adapt attack strategy

- Can handle at most $5 \frac{TB}{16} \text{ lists}/5 = 2^{36}$ entries per list.

- A straightforward implementation would use lists of size $2^{36}$: clamp 4 bits during list generation; this leads to $2^{36}$ values for each of the 16 lists (as we can generate at most $2^{40}$ elements per list).

- We expect to run the attack 256.5 times until we find a collision.
Attack in two phases

Idea

- First phase: Figure out which clamping constants yield collision
- Second phase: Compute matrix positions yielding collision
- During phase one we do not have to store positions of entries
- On each level compress entries to shortest possible representation
Attack in two phases

![Graph showing the relationship between bytes per entry and level with two lines: one for positions and another for compressed value.](image-url)
Attack in two phases

Idea

- First phase: Figure out which clamping constants yield collision
- Second phase: Compute matrix positions yielding collision
- During phase one we do not have to store positions of entries
- On each level compress entries to shortest possible representation
  - Level 0: 5 bytes (positions only)
  - Level 1: 10 bytes (positions only)
  - Level 2: 13 bytes (values only)
  - Level 3: 9 bytes (values only)
- Use lists of size $2^{37}$
- Clamp 3 bits through precomputation
- This leaves 4 bits “uncontrolled” $\rightarrow$ 16.5 repetitions expected
Attack Strategy

\[ 1152 \text{ GB} + 1664 \text{ GB} + 2560 \text{ GB} = 5376 \text{ GB} \]
Our hardware

Cluster of 8 nodes:

▶ Intel Core 2 Quad Q6600 CPU, 2.40 GHz
▶ 8 GB of RAM per node
▶ about 680 GB accessible mass storage
▶ connected via switched Gigabit Ethernet
Finding the bottleneck(s)
Parallelization

goals: L_{0,0} = 0,1; L_{0,1} = 0,1; L_{0,2} = 2,3; L_{0,3} = 2,3; L_{0,4} = 4,5; L_{0,5} = 4,5; L_{0,6} = 6,7; L_{0,7} = 6,7; L_{0,8} = 0,1,2,3; L_{0,9} = 0,1,2,3; L_{0,10} = 4,5,6,7; L_{0,11} = 4,5,6,7; L_{0,12} = 0,1,2,3; L_{0,13} = 0,1,2,3; L_{0,14} = 4,5,6,7; L_{0,15} = 4,5,6,7

L_{1,0} = 0,1,2,3; L_{1,1} = 0,1,2,3; L_{1,2} = 4,5,6,7; L_{1,3} = 4,5,6,7; L_{1,4} = 0,1,2,3,4,5,6,7; L_{1,5} = 0,1,2,3,4,5,6,7; L_{1,6} = 0,1,2,3,4,5,6,7; L_{1,7} = 0,1,2,3,4,5,6,7

L_{2,0} = 0,1,2,3,4,5,6,7; L_{2,1} = 0,1,2,3,4,5,6,7; L_{2,2} = 0,1,2,3,4,5,6,7; L_{2,3} = 0,1,2,3,4,5,6,7

L_{3,0} = 0,1,2,3,4,5,6,7; L_{3,1} = 0,1,2,3,4,5,6,7

L_{4,0} = 0,1,2,3,4,5,6,7

Final merge
Parallelization

- Split fractions further into 512 parts of 640 MB each (presort, according to 9 bits)
- Sort and merge parts independently in memory
- Pipeline
  - Loading from hard disk into memory
  - Sorting of two parts
  - Merging of previously sorted parts
- Requires 6 parts in memory at the same time (3.75 GB)
Parallelization

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- Requires 6 parts in memory at the same time (3.75 GB)
- Two blocks of operations:
  - Load, Sort, Merge, Send
  - Receive, Presort, Store
Timing Results

- Timings for phase 1:
  - Computation of list $L_{3,0}$: $\sim 32$ h (once)
  - Computation of list $L_{2,2}$: $\sim 14$ h (once)
  - Computation of list $L_{2,3}$: $\sim 14$ h (exp. $16.5 \times$)
  - Computation of list $L_{3,1}$: $\sim 4$ h (exp. $16.5 \times$)
  - Check for collision in $L_{3,0}$ and $L_{3,1}$: $\sim 3.5$ h (exp. $16.5 \times$)

- Expected time for phase 1: $32 + 14 + 16.5 \cdot (14 + 4 + 3.5) = 400.7$ h or 17 days

- Time for phase 2: $\sim 33$ h per half-tree, in total $\sim 66$ h

- Expected time in total: $\sim 19.5$ days.
Result

We already found a solution in step one after only five iterations!
In total the attack took 7 days, 23 hours and 53 minutes.

The result:

734, 15006, 20748, 25431, 33115, 46670, 50235, 51099, 70220, 76606,
89523, 90851, 99649, 113400, 118568, 126202, 144768, 146047, 153819,
163606, 168187, 173996, 185420, 191473 198284, 207458, 214106,
223080, 241047, 245456, 247218, 261928, 264386, 273345, 285069,
294658, 304245, 305792, 318044, 327120, 331742, 342519, 344652,
356623, 364676, 368702, 376923, 390678
Further information


Cluster: http://www.win.tue.nl/cccc/

Code: http://www.polycephaly.org/fsbdelay/
(available under public domain)